Physics 618, Spring 2020

March 20, 2020

Second Test

Last time: projective rep$^{\underline{n}}$'s

some aspects of Q.M.

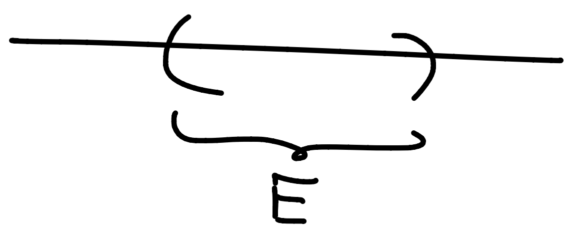
Born Rule:

State ρ pos. trace class, $\mathrm{Tr}(\rho) = 1$

Observable $\mathcal{O}$ self. adj.

on $\mathcal{H}$

$E \subset \mathbb{R}$



$$P_{\rho,\mathcal{O}}(E) = \mathrm{Tr}_{\mathcal{H}}\left(\rho\, P_{\mathcal{O}}(E)\right)$$

Symmetry in Q.M.

1-1 map

$s_1 : \mathcal{S} \longrightarrow \mathcal{S}$

$s_2 : \mathcal{O} \longrightarrow \mathcal{O}$

Preserves prob's.

$$\mathrm{Tr}\, s_1(\rho)\, P_{s_2(\mathcal{O})}(E) = \mathrm{Tr}\, \rho\, P_{\mathcal{O}}(E)$$

Such pairs form a group

Aut (QM).

Reduce this to a map

$s : \mathcal{S}^{pure} \longrightarrow \mathcal{S}^{pure}$

Preserves overlaps

$$\mathcal{S}^{pure} = \left\{ \text{Rank one projectors} \quad P = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} \quad \psi \in \mathcal{H} \right\}$$

$$\mathrm{Tr}\left(P_{\psi_1} P_{\psi_2}\right) = \frac{|\langle\psi_1|\psi_2\rangle|^2}{\langle\psi_1|\psi_1\rangle\langle\psi_2|\psi_2\rangle}$$

$$\mathcal{S}^{pure} = \mathbb{P}\mathcal{H}$$

If $\mathcal{H} = \mathbb{C}^{N+1}$ then $\mathcal{S}^{pure} = \mathbb{C}\mathbb{P}^N$

$$\vartheta(\ell_{\psi_1}, \ell_{\psi_2}) = \left(\cos\frac{d(\ell_1,\ell_2)}{2}\right)^2$$

$$\mathbb{P}\mathcal{H} = \left\{\begin{array}{c}\text{set of lines}\\ \text{through origin}\end{array}\right\}$$

$$= \left\{\begin{array}{c}\text{set of rank one}\\ \text{Projectors}\end{array}\right\}$$

$\mathrm{Aut}(QM) =$ Isometry group of $\mathbb{C}\mathbb{P}^N$ for

$d =$ Fubini-Study metric

Example: $\mathcal{H} = \mathbb{C}^2$ 1 Qbit

$$\rho = a + \vec{b}\cdot\vec{\sigma}$$

$\mathbb{1}, \vec{\sigma}$ span all 2×2 complex matrices.

$$\rho > 0 \iff (\psi, \rho\psi) \geq 0 \ \forall \psi$$

$$\rho^+ = \rho \iff a, \vec{b} \text{ real.}$$

Positivity ? e.v.'s ?

$$\vec{b}\cdot\vec{\sigma} \sim \begin{pmatrix} |\vec{b}| & \\ & -|\vec{b}| \end{pmatrix} \quad \checkmark$$

$\vec{b}\cdot\vec{\sigma}$ Hermitian for $\vec{b}$ real

$$\mathrm{Tr}(\vec{b}\cdot\vec{\sigma}) = 0$$

$$\vec{b}\cdot\vec{\sigma} \sim \begin{pmatrix} \lambda & \\ & -\lambda \end{pmatrix}$$

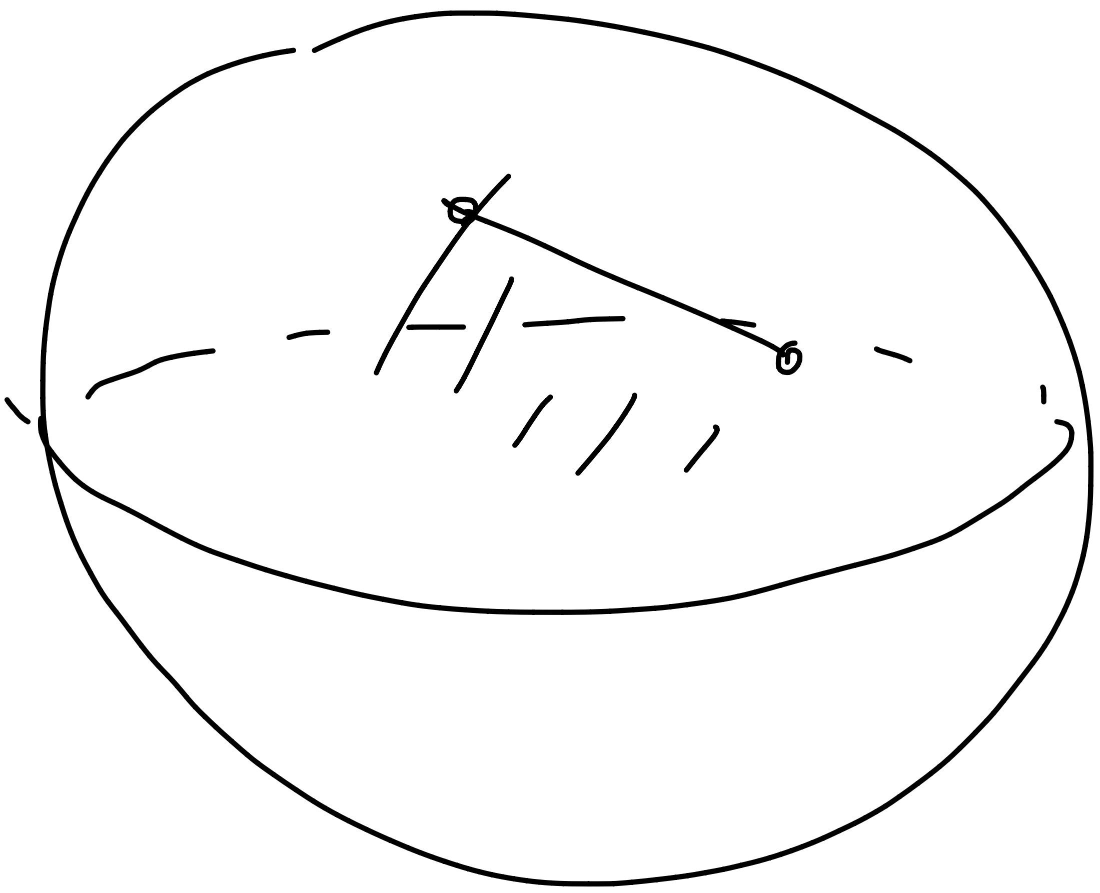
$$(\vec{b}\cdot\vec{\sigma})^2 = \vec{b}^2 = \lambda^2$$

$a + \vec{b}\cdot\vec{\sigma}$ e.v.'s $a \pm |\vec{b}|$

$\Longrightarrow \quad a \geq |\vec{b}|$

$\mathrm{Tr}(\rho) = 1 \Longrightarrow a = \frac{1}{2}$

$$\rho = \frac{1}{2}\left(\mathbb{1} + \vec{x}\cdot\vec{\sigma}\right) \qquad \|\vec{x}\| \leq 1$$

$\mathcal{S} =$  $\subset \mathbb{R}^3$

Extremal points

$$\mathcal{S}^{\text{Pure}} = S^2$$

$$\rho = \frac{1}{2}\left(\mathbb{1} + \hat{n}\cdot\vec{\sigma}\right) \qquad \hat{n}^2 = 1$$

$$\rho = \frac{1}{2}(\mathbb{1} + \hat{n}\cdot\vec{\sigma}) \qquad \hat{n}^2 = 1$$

$$\rho^2 = \rho \qquad \operatorname{Tr}\rho = 1$$

$$\Rightarrow \rho \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\rho = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$$ for some nonzero $\psi \in \mathbb{C}^2$

$\|\psi\| = 1$

$$\psi = u\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad u \in SU(2)$$

$$\overset{\text{Euler angle}}{=} \begin{pmatrix} e^{-i\frac{\psi+\phi}{2}}\cos\frac{\theta}{2} \\ e^{-i\frac{\psi-\phi}{2}}\sin\frac{\theta}{2} \end{pmatrix}$$

$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$$|\psi\rangle\langle\psi| = \frac{1}{2}(\mathbb{1} + \hat{n}\cdot\vec{\sigma})$$

$$S^3 \xrightarrow{\pi} S^2 \ni \hat{n}$$ Hopf fibration

$$S^3 = \{\psi \in \mathbb{C}^2 \mid \|\psi\|^2 = 1\}$$

$\pi^{-1}(\hat{n})$ is a $U(1)$ torsor

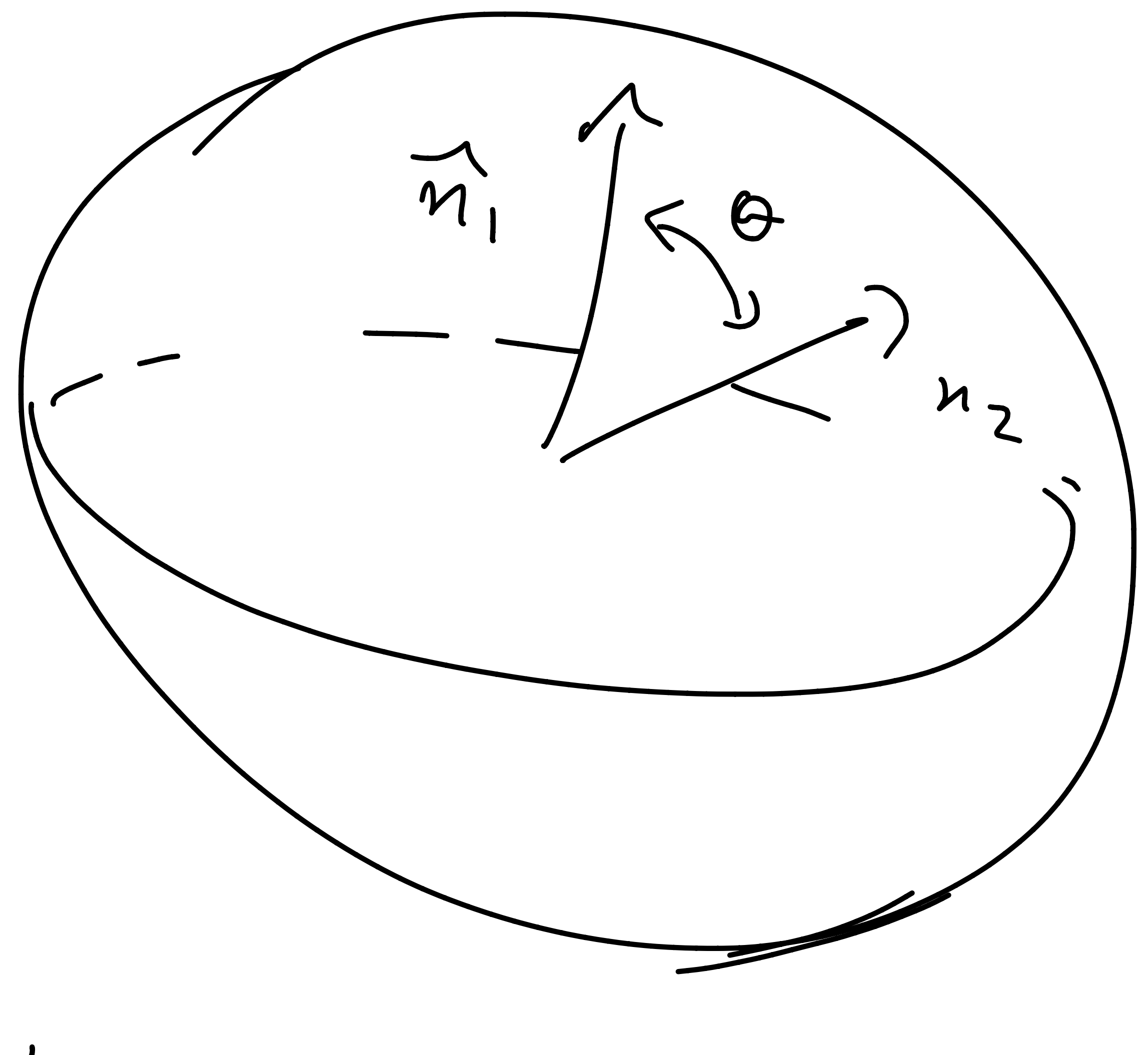
phase ambiguity in ψ

<u>Overlap</u>

$$\mathrm{Tr}\, P_{\hat{n}_1} P_{\hat{n}_2} = \frac{1}{2}(1 + \hat{n}_1 \cdot \hat{n}_2)$$

$$= \cos^2 \frac{\theta}{2}$$

θ = angle between $\hat{n}_1$ and $\hat{n}_2$



$\mathrm{Aut}(QM)$ = Isometry group of S^2
$= O(3)$

Next time:

Wigner's Theorem.

Awkward to describe symmetries in terms of isometries of the Fubini–Study metric on $\mathbb{P}\mathcal{H}$.

Replaces this by the group of unitary and anti-unitary operators on $\mathcal{H}$.

This linearizes the description.

A positive operator is a fortiori Hermitian

$A \geq 0$ then $A = B^\dagger B$ for some B

$B^\dagger B \geq 0$ $\quad (\psi, B^\dagger B \psi) = \|B\psi\|^2 \geq 0$

$$(\psi, A\psi) \geq 0$$

$$(\psi_1 + \lambda \psi_2, A(\psi_1 + \lambda\psi_2)) \geq 0$$

$$(\psi, A\psi)^* = (\psi, A\psi)$$

$$\parallel$$

$$(A\psi, \psi) = (\psi, A^\dagger \psi)$$

$$(e_i + e_j, A(e_i + e_j))$$

$$= (e_i + e_j, A^\dagger(e_i + e_j))$$

$\Rightarrow (e_j, Ae_i) + (e_i, Ae_j) =$ Same w/ $A \to A^\dagger$

Do the same thing with

$e_i + \sqrt{-1}\, e_j$ Combine the two eqs

$$(e_i, A e_j) = (e_i, A^\dagger e_j)$$

Wigner's Theorem:

$U(\mathcal{H})$ = group of unitary ops

$\mathbb{C}$- linear ops u s.t.

$$\|u\psi\| = \|\psi\| \quad \forall \psi \in \mathcal{H}$$

Set of anti-unitary op's.

$\mathbb{C}$-anti linear : $A(\psi_1+\psi_2) = A\psi_1 + A\psi_2$

$z \in \mathbb{C}$ $\qquad A(z\psi) = z^* A(\psi)$

"anti"

<u>anti-unitary</u> : a is $\mathbb{C}$-antilinear

and $\|a\psi\| = \|\psi\|$ for all $\psi \in \mathcal{H}$.

$\mathrm{Aut}(\mathcal{H})$ = group of unitary <u>and</u> anti-unitary operators.

$u_1, u_2 \in U(\mathcal{H})$

$u_1 \circ u_2 \in U(\mathcal{H})$

$a_1 \circ u_2$ anti unitary

$u_1 \circ a_2$ " "

$a_1 \circ a_2$ unitary

Act on the set of pure states

$P \longrightarrow u P u^{+}$ $\quad P = |\psi\rangle\langle\psi|$

$|\psi\rangle \to u|\psi\rangle$

$P \longrightarrow a P a^{+}$ $\leftarrow$

$$a a^{\dagger} = 1$$

$$u u^{\dagger} = 1$$

This preserves overlap function

$$o(P_1, P_2) = \mathrm{Tr}\, P_1 P_2$$

$$\longrightarrow \mathrm{Tr}\, u P_1 \cancel{u^{\dagger}}\, \cancel{u} P_2 u^{\dagger} = \mathrm{Tr} P_1 P_2$$

$$1 \to U(1) \longrightarrow \mathrm{Aut}(\mathcal{H}) \xrightarrow{\pi} \mathrm{Aut}(QM) \to 1$$

$\mathrm{Aut}(\mathcal{H})$ = group of unitary + antiunitary

$\mathrm{Aut}(QM)$ = Isom of F-S.

<u>Two points:</u>

(1) π is surjective.

(2) kernel is $U(1)$

$$\{ u = e^{i\theta} \cdot \mathbb{1} \} = \ker(\pi)$$

$$P \longrightarrow P$$